COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 46, No. 2 1997

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Periodic β-expansions for Certain Classes of Pisot Numbers

by

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(Received July 10, 1997)

Abstract. We give a characterization of numbers having periodic β-expansions where β belongs to certain classes of Pisot numbers: when β satisfies the equation $\beta^2=n\beta+1$ with $n\geq 1$ and when β satisfies the equation $\beta^2=n\beta-1$ with $n\geq 3$.

1. Introduction

Let $\beta>1$ be a real number, and let T_β be the β-transformation of the unit interval $[0, 1)$ given by $T_\beta x=\beta x \pmod 1$. The oribit of a real number $0\leq x<1$ by T_β, $Orb_\beta(x)=\{T_\beta^n(x)\}_{n\geq 0}$, gives the β-expansion of x, which is the following expansion:

$$x=\sum_{k=1}^{\infty} x_k\beta^{-k},$$

where x_k denotes the integer part of $\beta T_\beta^{k-1}x$. In the sequel, we simply write this $x=.x_1x_2x_3\cdots$. A real number $x\in[0, 1)$ is said to have an eventually periodic β-expansion with period p if there exist integers $k\geq 0$ and $p\geq 1$ such that $x=.x_1x_2\cdots x_k(x_{k+1}x_{k+2}\cdots x_{k+p})^\infty$. In particular, if we can choose $k=0$, then we say x has a periodic β-expansion with period p. Obviously, x has an eventually periodic β-expansion iff the set $Orb_\beta(x)$ is a finite set, and x has a periodic β-expansion with period p if and only if x is a periodic point of T_β with period p *i.e.* $T_\beta^p(x)=x$.

In this paper, we give a complete characterization of the set of numbers $0\leq x<1$ having periodic β-expansions, when β satisfies $\beta^2=n\beta+1$, $n\geq 1$, or $\beta^2=n\beta-1$, $n\geq 3$. We denote by $Per(\beta)$ the set of numbers $0\leq x<1$ having eventually periodic β-expansions, and by $P(\beta)$ having periodic β-expansions. Let $Q(\beta)$ be the smallest field extension of the field of rational numbers Q containing β. K. Schmidt proved [S] that if β is a Pisot number, then $Per(\beta)=Q(\beta)\cap[0, 1)$. Here β is said to be a *Pisot number* if β is an algebraic integer (>1) and all its Galois conjugates have modulus less than one. He also showed that for any β satisfying $\beta^2=n\beta+1$, $n\geq 1$, all rational numbers in $[0, 1)$ have periodic β-expansions. It is natural to ask the question whether, for any Pisot number β, every rational number has a periodic β-expansion or not. Our result gives a negative answer to this question as it says that no rational number in $(0, 1)$ has a periodic β-expansion whenever β is of the form $\beta^2=n\beta-1$, $n\geq 3$ (See Corollary 2).

For our purpose, the natural extension of T_β is useful. Y. Hara and Sh. Ito

[HI] considered, for quadratic irrational numbers β, expansions of real numbers in $[0, 1)$ by *modified* β-expansions in certain sense, and characterized $Q(\beta)$ in terms of eventually periodic orbits of a *modified* β-transformation and its natural extension. If β is the root of a polynomial $M(X)=X^2-nX+1$, this expansion is just the β-expansion and, especially, this gives the characterization of our $P(\beta)$. Also, Sh. Ito and H. Tachii [HT] considered *modified* continued fractions for quadratic numbers β and showed that $Q(\beta)$ is the set of periodic points for the modified continued fraction for β by using its natural extension. Indeed, our result concerns with "section" of their result. Because we need a natural extension which maps x in $Q(\beta)$ and its conjugate $\bar{x}$ to $T_\beta x$ and its conjugate to $\overline{T_\beta x}$, respectively, we cannot use the natural extension given by K. Dajani, C. Kraaicamp and B. Solomyak [DKS]. Sections of natural extensions considered in [HI] and [IT] turn out to be very useful for our problem.

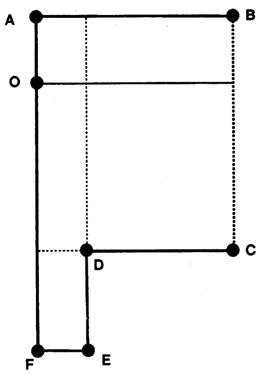
In §2, we state our main results. In section 3, for Pisot numbers β satisfying $\beta^2=n\beta+1$ $(n\geq 1)$ or $\beta^2=n\beta-1$ $(n\geq 3)$, we construct the natural extension of T_β and characterize the periodic points of T_β by the natural extension. In the penultimate section, we give proofs of main results. In the final section, we give some remarks on the length of periods of β-expansions which we considered.

Our thanks are due to Professor Y. Ito and Professor H. Nakada for their many helpful suggestions and encouragement.

2. Main results

In this paper, we construct a natural extension of the measurable dynamical system $([0, 1), \mathscr{A}, \mu_\beta, T_\beta)$, where $\mathscr{A}$ denotes the σ-algebra of Borel sets of $[0, 1)$ and μ_β denotes the invariant measure for β-transformation T_β, and give explicit descriptions for elements of $P(\beta)$ for β of the form $\beta^2=n\beta+1$ for some $n\geq 1$ or $\beta^2=n\beta-1$ for some $n\geq 3$.

Let $\beta>1$ be the Pisot number, which satisfies the equation $\beta^2=n\beta+1$ for some $n\geq 1$, and define $\bar{\mathscr{D}}_+ :=([0, \frac{1}{\beta})\times[-\beta, 1])\cup([\frac{1}{\beta}, 1)\times[1-\beta, 1])$.
(*cf.* Figure 1).



$O=(0,0)$, $A=(0,1)$, $B=(1,1)$, $C=(1,1-\beta)$,

$D=\left(\frac{1}{\beta}, 1-\beta\right)$, $E=\left(\frac{1}{\beta}, -\beta\right)$, $F=(0,-\beta)$

Figure 1.

We denote by $\bar{\mathscr{A}}_+$ the σ-algebra of Borel sets of $\bar{\mathscr{D}}_+$, and by $\bar{\mu}_+$ the normalized Lebesgue measure on $\bar{\mathscr{D}}_+$. Finally, we define the map $\bar{\mathscr{T}}_+$ on the space $\bar{\mathscr{D}}_+$ by

$$\bar{\mathscr{T}}_+(x, y) := (T_\beta x, \bar{\beta} y - [\beta x]),$$

where $\bar{\beta}$ is the Galois conjugate of β, *i.e.* $\bar{\beta} = -\frac{1}{\beta}$.

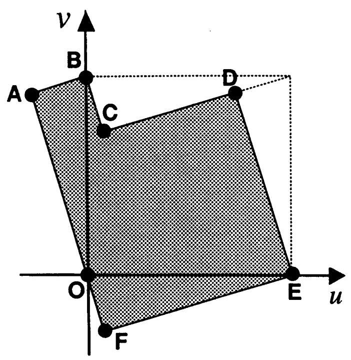
THEOREM 1. *The dynamical system* $(\bar{\mathscr{D}}_+, \bar{\mathscr{A}}_+, \bar{\mu}_+, \bar{\mathscr{T}}_+)$ *is a natural extension of the measurable dynamical system* $([0, 1), \mathscr{A}, \mu_\beta, T_\beta)$.

This is essentially due to [HI], this means the system is a section of maps $\{T_j\}$ considered in [HI].

Next, let q be a natural number. We denote by $D_+(q)$ the image of $(\bar{\mathscr{D}}_+)$ by the linear transformation induced from the matrix M_+:

$$M_+ := \frac{q}{\beta + \beta^{-1}} \begin{pmatrix} \beta & \beta^{-1} \\ 1 & -1 \end{pmatrix},$$

(*cf.* Figure 2).



$$O = (0, 0), \quad A = \left(-\frac{q}{\beta + \beta^{-1}}, \frac{q\beta}{\beta + \beta^{-1}} \right), \quad B = (0, q), \quad C = \left(\frac{q}{\beta(\beta + \beta^{-1})}, \frac{q(\beta + \beta^{-1} - 1)}{\beta + \beta^{-1}} \right),$$

$$D = \left(\frac{q(\beta + \beta^{-1} - 1)}{\beta + \beta^{-1}}, \frac{q\beta}{\beta + \beta^{-1}} \right), \quad E = (q, 0), \quad F = \left(\frac{q}{\beta(\beta + \beta^{-1})}, \frac{q}{\beta + \beta^{-1}} \right)$$

Figure 2.

Our first main result is as follows.

THEOREM 2. *Let* $n \geq 1$ *be a natural number, and let* β *be a Pisot number of the form* $\beta^2 = n\beta + 1$. *For any natural number* q, *set*

$$P_+(\beta, q) := \left\{ \frac{1}{q}(u + v\beta^{-1}) \,|\, (u, v) \in \mathscr{D}_+(q) \cap Z^2 \right\}.$$

Then,

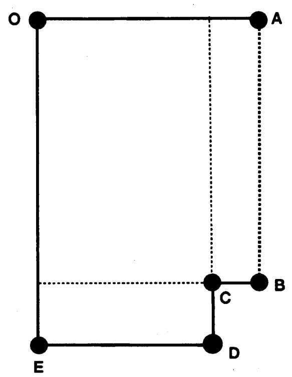
$$P(\beta)=\bigcup_{q\in N} P_+(\beta, q)\,.$$

If x is a rational number, then we regard x itself as its Galois conjugate. Thus, we can obtained the following result of Schmidt in [S] as a corollary.

COROLLARY 1. *For β of the form $\beta^2=n\beta+1$, $n\geq 1$, every rational number in* $[0, 1)$ *has a periodic β-expansion.*

This corollary should be compared with corollary 2 given later.

Now, let β be a Pisot number, with $\beta^2=n\beta-1$ for some $n\geq 3$, and define $\mathscr{D}_- := ([0, 1-\frac{1}{\beta}]\times[-\beta, 0])\cup([1-\frac{1}{\beta}, 1)\times[1-\beta, 0])$.
(*cf.* Figure 3).



$$O=(0,0)\,,\quad A=(1,0)\,,\quad B=(1,1-\beta)\,,\quad C=\left(1-\frac{1}{\beta}, 1-\beta\right),$$
$$D=\left(1-\frac{1}{\beta}, -\beta\right),\quad E=(0,-\beta)$$

Figure 3.

We denote by $\bar{\mathscr{A}}_-$ the σ-algebra of Borel sets of $\bar{\mathscr{D}}_-$, and by $\bar{\mu}_-$ the normalized Lebesgue measure on $\bar{D}_-$. Finally, we define the map $\bar{\mathscr{T}}_-$ on the space $\bar{D}_-$ by

$$\bar{\mathscr{T}}_-(x, y) := (T_\beta x, \bar{\beta}y-[\beta x])\,,$$

where $\bar{\beta}$ is the Galois conjugate of β, *i.e.* $\bar{\beta}=\frac{1}{\beta}$.

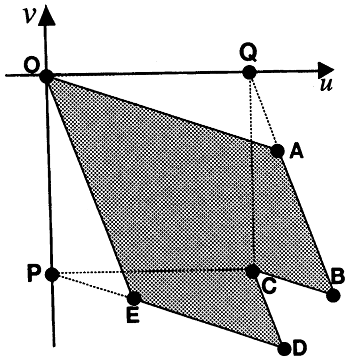
THEOREM 3. *The dynamical system $(\bar{\mathscr{D}}_-, \bar{\mathscr{A}}_-, \bar{\mu}_-, \bar{\mathscr{T}}_-)$ is a natural extension of the measurable dynamical system $([0, 1), \mathscr{A}, \mu_\beta, T_\beta)$.*

This is essentially due to [IT], this means the system is a section of T considered in [IT].

Let q be a natural number. We denote by $D_-(q)$ the image of $\bar{\mathscr{D}}$ by the linear transformation induced from the matrix M_-:

$$M_- := \frac{q}{\beta-\beta^{-1}}\begin{pmatrix} \beta & -\beta^{-1} \\ -1 & 1 \end{pmatrix},$$

(*cf.* Figure 4).



$$O=(0,0)\,,\quad A=\left(\frac{q\beta}{\beta-\beta^{-1}},\ -\frac{q}{\beta-\beta^{-1}}\right),\quad B=\left(\frac{q(\beta-\beta^{-1}+1)}{\beta-\beta^{-1}},\ -\frac{q\beta}{\beta-\beta^{-1}}\right),\quad C=(q,-q)\,,$$

$$D=\left(\frac{q\beta}{\beta-\beta^{-1}},\ -\frac{q(\beta-\beta^{-1}+1)}{\beta-\beta^{-1}}\right),\quad E=\left(\frac{q}{\beta-\beta^{-1}},\ -\frac{q\beta}{\beta-\beta^{-1}}\right),\quad P=(0,-q)\,,\quad Q=(q,0)$$

Figure 4.

Our second main result is as follows.

THEOREM 4. *Let $n \geq 3$ be a natural number, and let β be a Pisot number of the form $\beta^2 = n\beta - 1$. For any natural number q, set*

$$P_-(\beta, q) := \left\{\frac{1}{q}(u+v\beta^{-1}) \,|\, (u,v) \in \mathscr{D}_-(q) \cap Z^2\right\}.$$

Then,

$$P(\beta) = \bigcup_{q \in N} P_-(\beta, q)\,.$$

As a corollary, in contrast to Corollary 1, if β satisfies $\beta^2 = n\beta - 1$, we have the following:

COROLLARY 2. *Let β be a Pisot number of the form $\beta^2 = n\beta - 1$ for some $n \geq 3$. Then, no rational number in* $(0, 1)$ *has a periodic β-expansion.*

3. Natural extensions of β-transformations

In this section, we shall show that $\bar{\mathscr{T}}_\pm$ are natural extensions of measurable dynamical systems T_β for β's as before, respectively.

Note that if a real number x belongs to $Q(\beta)$, then $T_\beta(x)$ also belongs to $Q(\beta)$. If β is a quadratic algebraic integer, for any $x \in Q(\beta) = \{p_1 + p_2\beta^{-1} | p_1, p_2 \in Q\}$, we define a real number $\bar{x} := p_1 + p_2\bar{\beta}^{-1}$, where $\bar{\beta}$ is the Galois conjugate of β. In particular, if x is a rational number, then we define $\bar{x} = x$.

First, we consider the case that β_+ is of the form $\beta_+^2 = n\beta_+ + 1$ for some $n \geq 1$. We define a map $\hat{\mathscr{T}}_+ : [0, 1) \times R \to [0, 1) \times R$ by

$$\hat{\mathscr{T}}_+(x, y) := (T_{\beta_+}x, \bar{\beta}_+ y - [\beta_+ x]),$$

where $\bar{\beta}_+$ is the Galois conjugate of β_+, *i.e.* $\bar{\beta}_+ = -\frac{1}{\beta_+}$. It is easy to show the following lemma.

LEMMA 3.1. *For any real number* $x \in Q(\beta) \cap [0, 1)$,

$$\hat{\mathscr{T}}_+(x, \bar{x}) = (y, \bar{y}),$$

where $y = T_{\beta_+}x$, *and* $\bar{x}$ *and* $\bar{y}$ *are the Galois conjugates of* x *and* y, *respectively.*

Furthermore, we can prove the following proposition.

PROPOSITION 3.1. *A real number* $x \in Q(\beta) \cap [0, 1)$ *is a periodic point of* T_{β_+} *iff a point* $(x, \bar{x}) \in (Q(\beta) \cap [0, 1)) \times Q(\bar{\beta})$ *is a periodic point of* $\hat{\mathscr{T}}_+$.

Proof. Assume that a real number x is a periodic point of T_{β_+}. It is easy to see that x belongs to $Q(\beta) \cap [0, 1)$. Then it immediately follows from Lemma 3.1 that the point $(x, \bar{x})$ is a periodic point of $\hat{\mathscr{T}}_+$.

Converse is trivial. ∎

Since it is clear that $\bar{\mathscr{T}}_+$ is the restriction of $\hat{\mathscr{T}}_+$ to the space $\bar{\mathscr{D}}_+$, we can prove Theorem 1.

Proof of Theorem 1. It immediately follows from the definitions that $\bar{\mu}_+$ is an invariant measure of $\bar{\mathscr{T}}_+$. Furthermore, it can be easily seen that the map $\bar{\mathscr{T}}_+$ is a bijective map from $\bar{\mathscr{D}}_+ - \bar{X}_+$ to $\bar{\mathscr{D}}_+ - \bar{Y}_+$. Here $\bar{X}_+$ and $\bar{Y}_+$ are $\partial cl\bar{\mathscr{D}}_+ \cup \bigcup_{i=1}^{i=n-1} (\{\frac{i}{\beta_+}\} \times (1-\beta_+, 1))$ and $\partial cl\bar{\mathscr{D}}_+ \cup \bigcup_{i=1}^{i=n} ((0, 1) \times \{i - \beta_+\})$, which are null sets. The definition of $\bar{\mathscr{T}}_+$ implies that it is bi-measurable. Thus $(\bar{\mathscr{D}}_+, \bar{\mathscr{A}}_+, \bar{\mu}_+, \bar{\mathscr{T}}_+)$ is an automorphism. Define the projection map $\imath_+ : \bar{\mathscr{D}}_+ \to [0, 1)$ by $\imath_+(x, y) = x$. The definition directly implies $\imath_+^{-1}(A) \in \bar{\mathscr{A}}_+$ and $\bar{\mu}_+(\imath_+^{-1}(A)) = \mu_+(A)$ for any $A \in \mathscr{A}$. Thus it is an endomorphism from $\bar{\mathscr{D}}_+$ to $[0, 1)$. Furthermore, it can be easily seen that $\imath_+ \circ \bar{\mathscr{T}}_+ = T_{\beta_+} \circ \imath_+$. Since $|\bar{\beta}_+| < 1$, it is not so hard to see that $\{\imath_+^{-1}(\{x : x_1 = p\}) | p = 0, 1, \cdots, n+1\}$ is a generator for $\bar{\mathscr{T}}_+$, which implies the assertion of the theorem. ∎

Next, let β_- be a Pisot number satisfying $\beta_-^2 = n\beta_- - 1$ for some $n \geq 1$. We define the map $\hat{\mathscr{T}}_- : [0, 1) \times R \to [0, 1) \times R$ by

$$\bar{\mathscr{T}}_-(x, y) := (T_{\beta_-}x, \bar{\beta}_- y - [\beta_- x]),$$

where $\bar{\beta}_-$ is the Galois conjugate of β_-, *i.e.* $\bar{\beta}_- = \frac{1}{\beta_-}$.

It is easy to see that the following result is valid by using the same argument as in the first case.

LEMMA 3.2. *For any real number* $x \in Q(\beta) \cap [0, 1)$,

$$\hat{\mathscr{T}}_{-}(x, \bar{x}) = (y, \bar{y}),$$

where $y = T_{\beta_+}x$, *and* $\bar{x}$ *and* $\bar{y}$ *are the Galois conjugates of* x *and* y, *respectively.*

PROPOSITION 3.2. *A real number* $x \in Q(\beta) \cap [0, 1)$ *is a periodic point of* T_{β_-} *iff a point* $(x, \bar{x}) \in (Q(\beta) \cap [0, 1)) \times Q(\bar{\beta})$ *is a periodic point of* $\hat{\mathscr{T}}_{-}$.

Since it is clear that $\bar{\mathscr{T}}_{-}$ is the restriction of $\hat{\mathscr{T}}_{-}$ to the space $\bar{\mathscr{D}}_{-}$, we can prove Theorem 3.

Proof of Theorem 3. It immediately follows from the definitions that $\bar{\mu}_{-}$ is an invariant measure of $\bar{\mathscr{T}}_{-}$. Furthermore, it can be easily seen that the map $\bar{\mathscr{T}}_{-}$ is a bijective map from $\bar{\mathscr{D}}_{-} - \bar{X}_{-}$ to $\bar{\mathscr{D}}_{-} - \bar{Y}_{-}$. Here $\bar{X}_{-}$ and $\bar{Y}_{-}$ are $\partial cl\bar{\mathscr{D}}_{-} \cup (\{\frac{n-1}{\beta_-}\} \times (1-\beta_-, 0)) \cup \bigcup_{i=1}^{i=n-2} (\{\frac{i}{\beta_-}\} \times (-\beta_-, 0))$ and $\partial cl\bar{\mathscr{D}}_{-} \cup ((1-\beta_-, 0) \times \{1-n\}) \cup \bigcup_{i=-n+2}^{i=-1} ((0, 1) \times \{i\})$, which are null sets. The definition of $\bar{\mathscr{T}}_{-}$ also implies that it is bi-measurable. Thus $(\bar{\mathscr{D}}_{-}, \bar{\mathscr{A}}_{-}, \bar{\mathscr{T}}_{-})$ is an automorphism. Define the projection map $\iota_{-} : \bar{\mathscr{D}}_{-} \to [0, 1)$ by $\iota_{-}(x, y) = x$. The definition directly implies $\iota_{-}^{-1}(A) \in \bar{\mathscr{A}}_{-}$ and $\bar{\mu}_{-}(\iota_{-}^{-1}(A)) = \mu_{-}(A)$ for any $A \in \mathscr{A}$. Thus it is an endomorphism from $\bar{\mathscr{D}}_{-}$ to $[0, 1)$. Furthermore, it can be easily seen that $\iota_{-} \circ \bar{\mathscr{T}}_{-} = T_{\beta_-} \circ \iota_{-}$. Since $|\bar{\beta}_{-}| < 1$, it is not so hard to see that $\{\iota_{-}^{-1}(\{x : x_1 = p\}) \mid p = 0, 1, \cdots, n\}$ is a generator for $\bar{\mathscr{T}}_{-}$, which implies the assertion of the theorem. ∎

In the sequel, we use, without stating it explicitly, β to mean either β_+ or β_-, and $\bar{\mathscr{D}}$ to mean $\bar{\mathscr{D}}_{+}$ or $\bar{\mathscr{D}}_{-}$, respectively, and so on, whenever it is clear from the context which one is referred to.

4. Proofs of main results

In this section, we prove main theorems. Let q be a natural number, and we write

$$F(\beta, q) := \left\{ x = \frac{1}{q}(u + v\beta^{-1}) \,\middle|\, (u, v) \in Z^2 \right\} \subset Q(\beta).$$

LEMMA 4.1. *The set* $F(\beta, q) \cap [0, 1)$ *is an invariant set of the* β*-transformation* T_β.

Proof. We only show this lemma for β_+. Let $x \in F(\beta, q)$ be of the form $\frac{1}{q}(u + v\beta^{-1})$. If $x = .x_1x_2x_3 \cdots$, then

$$\begin{aligned} T_\beta x &= \beta x - [\beta x] \\ &= \frac{1}{\mathrm{q}}(u\beta + v) - x_1 \\ &= \frac{1}{q}\left((nu + v) + \frac{v}{\beta}\right) - x_1 \end{aligned}$$

$$=\frac{1}{q}\left((nu+v-qx_1)+\frac{v}{\beta}\right).$$

Therefore, $T_\beta x$ belongs to the set $F(\beta, q)\cap[0, 1)$. ■

Now, we can obtain the following fact, which gives a characterization of periodic points of T_β.

The following Proposition 4.1 is a special case of [HI] and [IT], because of Theorem 1 and Theorem 3. However, our systems consist of two dimensional maps and this fact makes the proof of our case simpler than that of theirs.

We also note that the idea of the box principle method (e.g. [S]) is clearer in our proof.

PROPOSITION 4.1. *A real number $x\in Q(\beta)\cap[0, 1)$ is a periodic point of T_β iff a point $(x, \bar{x})$ belongs to the region $\bar{\mathscr{D}}$.*

Proof. From Proposition 3.1 and Proposition 3.2, it is sufficient to show that a poing $(x, \bar{x})$ is a periodic point of $\hat{\mathscr{T}}$ iff the point $(x, \bar{x})$ belongs to the region $\bar{\mathscr{D}}$. Let q be a natural number such that $x\in F(\beta, q)$. Assume that a point $(x, \bar{x})$ belongs to the region $\bar{\mathscr{D}}$. It follows from Lemma 4.1 and the definition of the map $\hat{\mathscr{T}}$ that the orbit of $(x, \bar{x})$ under $\hat{\mathscr{T}}$ is contained in a set $(F(\beta, q)\times F(\bar{\beta}, q))\cap\bar{\mathscr{D}}$. Because $(F(\beta, q)\times F(\bar{\beta}, q))\cap\bar{\mathscr{D}}$ is a finite set, the orbit $\{\hat{\mathscr{T}}^k(x, \bar{x})\}_{k\geq 0}$ is a finite set and the point $(x, \bar{x})$ is a eventually periodic point of $\hat{\mathscr{T}}$. Note that $\hat{\mathscr{T}}=\bar{\mathscr{T}}$ is a bijective map of the set $(F(\beta, q)\times F(\bar{\beta}, q))\cap\bar{\mathscr{D}}$. Therefore, the point $(x, \bar{x})$ is a periodic point of $\hat{\mathscr{T}}$. This argument shows us that the sufficient condition of the statement is true.

Next, define a map $F_{\bar{\mathscr{D}}}:=Q(\beta)\cap[0, 1)\times Q(\bar{\beta})\to R$ by

$$F_{\bar{\mathscr{D}}}:=\inf_{(x, y)\in\bar{\mathscr{D}}}\{|\bar{x}-y|\}.$$

This map gives a distance function between the point $(x, \bar{x})$ and the region $\bar{\mathscr{D}}$, and it is clear that $F_{\bar{\mathscr{D}}}(x, \bar{x})=0$ iff $x=\frac{1}{\beta}$ or $(x, \bar{x})\in\bar{\mathscr{D}}$. If $x=\frac{1}{\beta}$, then $x=.1000\cdots$ and does not have a periodic β-expansion.

Assume that $(x, \bar{x})\notin\bar{\mathscr{D}}$ and $x\neq\frac{1}{\beta}$. There exists a point $(x, y_0)\in\partial\bar{\mathscr{D}}$ such that $F_{\bar{\mathscr{D}}}(x, \bar{x})=|\bar{x}-y_0|$.

$$\begin{aligned}F_{\bar{\mathscr{D}}}(\hat{\mathscr{T}}(x, \bar{x}))&=\inf_{(T_\beta x, y)\in\bar{\mathscr{D}}}\{|\overline{T_\beta x}-y|\}\leq\mathrm{Dis}(\hat{\mathscr{T}}(x, \bar{x}), \hat{\mathscr{T}}(x, y_0))\\&=|\overline{T_\beta x}-(\bar{\beta}y_0-[\beta x])|\\&=|\bar{\beta}||\bar{x}-y_0|<F_{\bar{\mathscr{D}}}(x, \bar{x}).\end{aligned}$$

Where Dist$((x, y), (x', y'))$ denotes the Euclid distance between (x, y) and (x', y').

Therefore, the point $(x, \bar{x})$ cannot be a periodic point of $\hat{\mathscr{T}}$ if $F_{\bar{\mathscr{D}}}(x, \bar{x})>0$. This argument shows that the necessity of the condition is true. ■

From Proposition 4.1 and Theorem 1 (and Figure 1), we can prove Corollary

1, which is claimed by Schmidt [S].

Proof of Corollary 1. Theorem 1 shows that the region $\bar{\mathscr{D}}_+$ contains the set $\{(x, x) \in R^2 \mid 0 \leq x < 1\}$. Because $\bar{x} = x$ for any rational number x, we have the desired result. ■

Also, we can prove Corollary 2 (*cf.* Figure 4).

Proof of Corollary 2. Theorem 2 shows that the region $\bar{\mathscr{D}}_-$ does not contain the set $\{(x, x) \in R^2 \mid 0 < x < 1\}$. Because $\bar{x} = x$ for any rational number x, we have the desired result. ■

Proposition 4.1 also implies our main theorems.

Proof of Theorem 2. Let q be a fixed natural number. It follows from Proposition 4.1 that a real number $x \in F(\beta, q)$ is a periodic point of T_β iff a point $(x, \bar{x}) \in \bar{\mathscr{D}}_+$. Note that x is of the form $\frac{1}{q}(u + v\beta^{-1})$ and $\bar{x}$ is of the form $\frac{1}{q}(u - v\beta)$. This fact shows that $(x, \bar{x}) \in \bar{\mathscr{D}}_+$ iff $(u, v) \in M_+(\bar{\mathscr{D}}_+) = \mathscr{D}_+(q)$. Finally, $Q(\beta) \cap [0, 1) = \bigcup_{q \in N} F(\beta, q) \cap [0, 1)$ implies the assertion of the theorem. ■

Proof of Theorem 4. Let q be a fixed natural number. It follows from Proposition 4.1 that a real number $x \in F(\beta, q)$ is a periodic point of T_β iff a point $(x, \bar{x}) \in \bar{\mathscr{D}}_-$. Note that x is of the form $\frac{1}{q}(u + v\beta^{-1})$ and $\bar{x}$ is of the form $\frac{1}{q}(u + v\beta)$. This fact shows that $(x, \bar{x}) \in \bar{\mathscr{D}}_-$ iff $(u, v) \in M_-(\bar{\mathscr{D}}_-) = \mathscr{D}_-(q)$. Finally, $Q(\beta) \cap [0, 1) = \bigcup_{q \in N} F(\beta, q) \cap [0, 1)$ implies the assertion of the theorem. ■

5. The length of periods of points of $P(\beta, q)$

In this section, we deal with periods of points of the set $P(\beta, q)$. Let q be a natural number, and, for a real number $x \in P(\beta, q)$, write

$$p_\pm(x) = p_\pm(x, q) := \min_{l \geq 1} \{l \mid \bar{\mathscr{T}}^l_{\beta_\pm}(x, \bar{x}) = (x, \bar{x})\},$$

which we call the (minimal) period of x. We set

$$\hat{p}_\pm := \max_{x \in P_\pm(\beta_\pm, q)} p_\pm(x),$$

and call it the maximal periodicity for $P_\pm(\beta, q)$, respectively. Let $(u, v) \in \mathscr{D}(q)$ for some natural number q. What is the period of a real number $x = \frac{1}{q}(u + v\beta^{-1})$? Following examples show us that there exists $x \in P(\beta, q)$ such that $p(x)$ is a divisor of $\hat{p}$.

EXAMPLE 1. *Let $\beta > 1$ satisfy $\beta^2 = 5\beta + 1$ i.e. $\beta = (5 + \sqrt{29})/2$, and $q = 7$. Then, we have $\hat{p}_+ = 6$, but a real number $\frac{1}{7}(3 + 5\beta^{-1}) = (-19 + 5\sqrt{29})/14$ has a periodic β-expansion with period* 3.

EXAMPLE 2. *Let $\beta > 1$ satisfy $\beta^2 = 5\beta - 1$ i.e. $\beta = (5 + \sqrt{21})/2$, and $q = 3$. Then, we have $\hat{p}_- = 3$, but a real number $\frac{1}{3}(1 - 2\beta^{-1}) = (-4 + \sqrt{21})/3$ has a periodic β-*

expansion with period 1.

In the sequel, we discuss some relations among $\hat{p}$ and $p(x)$. Let $Z[\lambda]$ be the ring of polynomials in λ with integral coefficients, $Z_+[\lambda]$ the cone of polynomials with nonnegative coefficients, and set $(Z[\lambda])_+ = Z[\lambda] \cap R_+$. Then we obtain the following propositions.

PROPOSITION 5.1. *Let $\beta > 1$ satisfy the equation $\beta^2 = n\beta + 1$ for some $n \geq 1$, and let q be a natural number.*

Then, for a real number $x \in P(\beta, q)$, $p(x)$ is a divisor of $\hat{p}$, and is the minimal natural number l for which the real number $x \cdot (1-\beta^{-1}) \in (Z[\beta^{-1}])_+$.

PROPOSITION 5.2. *Let $\beta > 1$ satisfy the equation $\beta^2 = n\beta - 1$ for some $n \geq 3$, and let q be a natural number.*

Then, for a real number $x \in P(\beta, q)$, $p(x)$ is a divisor of $\hat{p}$. Furthermore, if there exists the minimal natural number l for which the real number $x \cdot (1-\beta^{-l}) \in Z_+[\beta^{-1}]$, then $p(x) = l$.

We consider dynamical systems, introduced by K. Schmidt [S], which are isomorphic to systems $(\bar{\mathscr{T}}, \bar{\mathscr{D}} \cap (Q(\beta) \times Q(\bar{\beta})))$. We can prove the propositions by using these dynamics.

Firstly, let $\mathscr{S}_\pm$ be maps of $\mathscr{D}_\pm \cap Z^2$ determined by the relations $\mathscr{S}_\pm \circ M_\pm = M_\pm \circ \bar{\mathscr{T}}_\pm$, respectively.

$$i.e.\quad \mathscr{S}_\pm \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} n & 1 \\ \pm 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - q \cdot [\beta x] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is clear, by the definition of $\mathscr{S}_\pm$, that systems $(\mathscr{S}_\pm, \mathscr{D}_\pm \cap Z^2)$ are isomorphic to $(\bar{\mathscr{T}}_\pm, \bar{\mathscr{D}}_\pm \cap (Q(\beta_\pm) \times Q(\bar{\beta}_\pm)))$, respectively.

Next, define the map $\pi_+ : \mathscr{D}_+ \cap Z^2 \to (Z/qZ)^2$ by (*cf.* Figure 5)

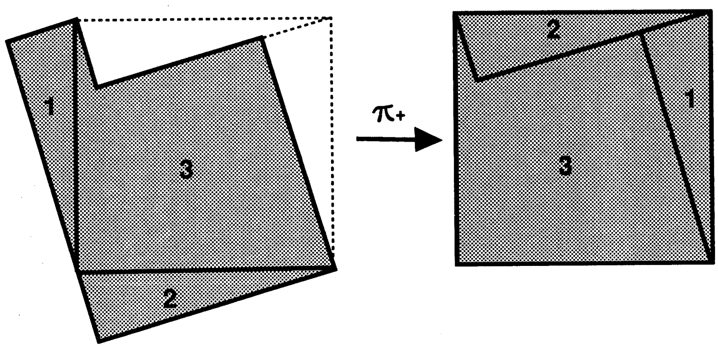


Figure 5.

$$\pi_+\begin{pmatrix}u\\v\end{pmatrix}:=\begin{cases}\begin{pmatrix}u+q\\v\end{pmatrix}, & \text{if } u<0\\ \begin{pmatrix}u\\v+q\end{pmatrix}, & \text{if } v<0\\ \begin{pmatrix}u\\v\end{pmatrix}, & \text{otherwise}\end{cases},$$

and the map $\pi_-:=\mathscr{D}_-\cap Z^2\rightarrow(Z/qZ)^2$ by (*cf.* Figure 6)

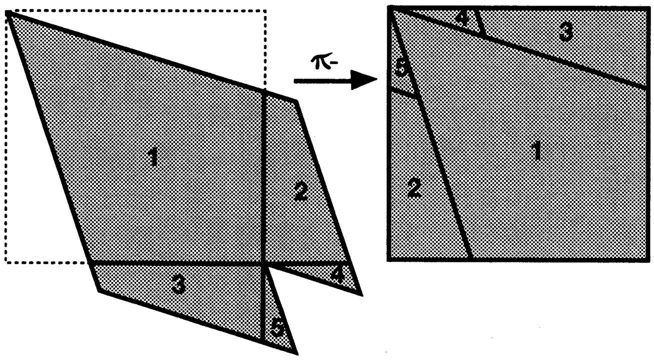


Figure 6.

$$\pi_-\begin{pmatrix}u\\v\end{pmatrix}:=\begin{cases}\begin{pmatrix}u-q\\v+q\end{pmatrix}, & \text{if } u\geq q \text{ and } v\geq -q\\ \begin{pmatrix}u-q\\v+2q\end{pmatrix}, & \text{if } u\geq q \text{ and } v<-q\\ \begin{pmatrix}u\\v+2q\end{pmatrix}, & \text{if } u<q \text{ and } v<-q\\ \begin{pmatrix}u\\v+q\end{pmatrix}, & \text{otherwise}\end{cases}.$$

Furthermore, let $A_{\beta_\pm}$ be the 2×2 integral matrices

$$A_{\beta_\pm}:=\begin{pmatrix}n & 1\\ \pm 1 & 0\end{pmatrix},$$

and let $A(\beta_\pm, q)$ be the automorphism of $(Z/aZ)^2$ induced by the matrices $A_{\beta_\pm}$, respectively.

LEMMA 5.1. *The maps $\bar{\mathscr{T}}_\pm$ are conjugate to the maps $A(\beta_\pm, q)$ by $\pi_\pm$, respectively.*

$$\textit{i.e.}\quad \pi_\pm\circ\bar{\mathscr{T}}_\pm=A(\beta_\pm, q)\circ\pi_\pm .$$

Proof. It is clear that π is a tilling map of the region $\mathscr{D}$. This implies the required result. ∎

Thus, we can conclude the following result, which describes the relation between the two dynamical systems.

PROPOSITION 5.3. *Dynamical systems* $(A(\beta_\pm, q), (Z/qZ)^2)$ *are isomorphic to systems* $(\bar{\mathscr{T}}_\pm, \bar{\mathscr{D}}_\pm)$*, respectively. Furthermore,*

$$\hat{p}=\max_{(s,t)\in(Z/qZ)^2}\min_{l\geq 1}\left\{l \,\middle|\, A(\beta, q)^l\begin{pmatrix} s \\ t \end{pmatrix}=\begin{pmatrix} s \\ t \end{pmatrix}\right\}.$$

Proof. This proposition immediately follows from definitions of $\pi_\pm$ and Lemma 5.1. ∎

Now, we give a simple observation for periodic points of $(A(\beta_\pm, q), (Z/qZ)^2)$. We denote by $ord(A(\beta, q))$ the minimal integer $l\geq 1$ such that $A(\beta, q)^l=id$. For $(s, t)\in(Z/qZ)^2$, set

$$p'(s, t)=\min_{l\geq 1}\left\{l \,\middle|\, A(\beta, q)^l\begin{pmatrix} s \\ t \end{pmatrix}=\begin{pmatrix} s \\ t \end{pmatrix}\right\}.$$

Note that the natural number $p'(s, t)$ is a divisor of $ord(A(\beta, q))$.

LEMMA 5.2.

$$\hat{p}=p'(1, 0)=ord(A(\beta, q)).$$

Proof. It is clear that $\hat{p}=ord(A(\beta, q))$. To complete the proof, it is sufficient to show that $p'(1, 0)=ord(A(\beta, q))$. For any $l\geq 1$, there exist natural numbers $\alpha(l)$, $\beta(l)$ such that $A^l_{\beta_\pm}=\alpha(l)A_{\beta_\pm}+\beta(l)id$. On the other hand, the shapes of the matrices $A_{\beta_\pm}$, show that $A^{p'(1,0)}_{\beta_\pm}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, $A^{p'(1,0)}_{\beta_\pm}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\alpha(p'(1, 0))\begin{pmatrix} 0 \\ 1 \end{pmatrix}+\beta(p'(1, 0))$, and $\alpha(p'(1, 0))\equiv 0$, $\beta(p'(1, 0))\equiv 1 \pmod{q}$. ∎

Now, return to the systems $(\bar{\mathscr{T}}_\pm, \bar{\mathscr{D}}_\pm)$. Proposition 5.3 and Lemma 5.2 directly imply the following lemma.

LEMMA 5.3. $\hat{p}=p\left((\pi\circ M)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, *and* $p(x)$ *is a divisor of* $\hat{p}$ *for* $x\in P(\beta, q)$.

Fix $q\in N$, and let $(u, v)\in\mathscr{D}\cap Z^2$, and let $x=\frac{1}{q}(u+v\beta^{-1})$. Lemma 5.3 and an easy calculation show that a real number $x\cdot(1-\beta^{-\hat{p}})$ has a finite β-expansion. *i.e.* there exist coefficients $(x_1x_2x_3\cdots)\in\{0, 1, \cdots, [\beta]\}^\infty$ such that

$$x\cdot(1-\beta^{-\hat{p}})=x_1\beta^{-1}+x_2\beta^{-2}+\cdots+x_{\hat{p}}\beta^{-\hat{p}}.$$

LEMMA 5.4. $p(x)=l$ *iff* l *is the minimal number, which is a divisor of* $\hat{p}$*, for which the real number* $x\cdot(1-\beta^{-1})$ *has a finite* β*-expansion.*

Proof. It follows from the fact that l is a divisor of $\hat{p}$, that

$$x_1\beta^{-1}+x_2\beta^{-2}+\cdots+x_{\hat{p}}\beta^{-\hat{p}}$$
$$=(x_1\beta^{-1}+x_2\beta^{-2}+\cdots+x_l\beta^{-l})(1+\beta^{-l}+\cdots+\beta^{-(\hat{p}-l)}).$$

Therefore, we have

$$x\cdot(1-\beta^{-l})=x\cdot\frac{1-\beta^{-\hat{p}}}{1+\beta^{-l}+\cdots+\beta^{-(\hat{p}-l)}}$$
$$=x_1\beta^{-1}+x_2\beta^{-2}+\cdots+x_l\beta^{-l}.$$

This shows that the necessity of the condition is true. We can see that the sufficiency is true by reversing the argument above. ■

Now, let $Fin(\beta)$ be the set of numbers $0\leq x<1$ having a finite β-expansion. It is clear that a real number x has a finite β-expansion iff the orbit of x under T_β eventually goes to zero. C. Frougny and B. Solomyak gave characterizations of the set $Fin(\beta)$ for certain classes of β as follows [FS];

THEOREM A1. (C. Frougny and B. Solomyak) *Let β be the positive root of the polynomial $M(X)=X^m-a_1X^{m-1}-a_2X-\cdots-a_m$, $a_i\in Z$, and $a_1\geq a_2\geq\cdots\geq a_m>0$. Then β is a Pisot number, and $Fin(\beta)=(Z[\beta^{-1}])_+$.*

THEOREM A2. (C. Frougny and B. Solomyak) *Let β be the positive root of the polynomial $M(X)=X^{m+1}-(t_1+1)X^m+(t_1-t_2)X^{m-1}+\cdots+(t_m-t_{m-1})X+(t_m-t_{m+1})$, with $t_1\geq t_2\geq\cdots\geq t_m>t_{m+1}>0$. Then β is a Pisot number, and $Z_+[\beta^{-1}]\subset Fin(\beta)$.*

Proof of Proposition 5.1. Obviously, the β is the positive root of polynomial $M(X)=X^2-nX-1$, and $a_1=n\geq a_2=1>0$. Thus, Theorem A1 shows that $Fin(\beta)=(Z([\beta^{-1}]))_+$. Therefore, Lemma 5.4 directly implies the assertion of the proposition. ■

Proof of Proposition 5.2. Obviously, the β is the positive root of polynomial $M(X)=X^2-nX+1$, and $t_1=n-1>t_2=n-2>0$. Thus, Theorem A2 shows that $Z_+[\beta^{-1}]\subset Fin(\beta)$. Therefore, the sufficiency of the condition of Lemma 5.4 directly implies the assertion of the proposition. ■

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